QUASI-STATIONARY FLOW OF A GAS FROM A CYLINDRICAL CONTAINER OF VARIABLE VOLUME

(KVAZISTATSIONARNOYE ISTECHENIE GAZA IZ TSILINDRICHESKOGO SOSUDA PEREMENNOGO OB'EMA)

PMM Vol.22, No.2, 1958, pp.279-285

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(Received 8 July 1958)

One considers the problem of the flow of gas through a small opening in the frontal wall of the cylindrical container, the rear wall of which is movable. While the gas is leaving the container the latter is being replenished with a new gas. This problem is of interest in the theory of gasoline engines, in the field of internal ballistics, etc.

1. Formulation of the problem. One introduces into a cylindrical container, the rear wall of which acts as a heavy piston displaceable under the action of expanding gases, a gas at temperature T_1 (this temperature could be the combustion temperature of the fuel) having no initial velocity. The gas streams from the container through a small opening in the frontal wall into a medium without a backpressure, so that at the location of the opening one obtains the critical flow and therefore the velocity of the gas at the exit equals the local velocity of sound (see the figure).

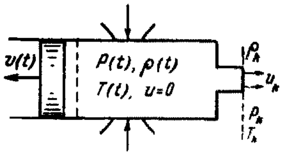

Fig. 1.

The greatest difficulty in the non-stationary problem here considered - the problem of the outflowing gas - consists in the intrinsic motion of the gas within the cylinder, inasmuch as the interaction of waves reflected, both from the piston and from the frontal wall, and propagating in the moving gas presents a formidable physical picture.

Thus, in the solution of the Lagrange problem as proposed by Love and Pidduck [1], the formulas relating to the second reflected wave become already so complicated that their practical usefulness becomes rather questionable.

Reprint Order No. PMM 35.

Here we shall adopt the following approximate formulation of the problem. We shall assume that after the propagation of the first few reflected waves a condition is being attained in the gas during which the basic state parameters, namely the pressure p, the density ρ and the absolute temperature T, vary little at a given instant of time from one point to another and remain functions of time t only; i.e. $p = p(t)$, $\rho = \rho(t)$ and $T = T(t)$.

If the diameter of the container is great by comparison with the opening of the exit and the mass of the piston sufficiently large by comparison with the mass of the gas within the container, then one may neglect the self-excited motion of the gas within the container, putting the velocity of the gas $u = 0$.

The layer of gas next to the piston has obviously the velocity equal to that of the piston $v(t)$. This circumstance can be taken into consideration in formulating the equations of motion and of energy, if instead of the mass m of piston one considers its induced mass M.

2. Equation of Energy. If during the time interval dt, $G_0 d\dot{\psi}$ kg of gas at temperature T_1 enters the container, then the amount of energy which is being conveyed along with the gas to the container equals

$$dE = c_v T_1 G_0 d\psi = \frac{RT_1}{k-1} G_0 d\psi \qquad \left(k = \frac{c_p}{c_v}\right)$$

where R is the gas constant, c_p the specific heat at a constant pressure, c_v the specific heat at a constant volume and ψ is a dimensionless quantity equal to the ratio of the amount of gas entering the container at time t to the total amount G_0 kg of gas entering the container.

The energy dE thus entering the container is expended in changing the internal energy dE_1 of gases within the container at the time t, on the energy dE_2 carried by the gases streaming through the opening and on imparting the kinetic energy dE_3 to the piston and adjacent layers of gas. We shall calculate all those energy contributions. If $G(t)$ denotes the mass flow of gases streaming through the opening and $\eta(t)$ is the relative expenditure of gases

$$\eta(t) = \frac{1}{G_0} \int_0^t G(t)\, dt \tag{2.1}$$

then at time t one finds in the container $G_0(\psi - \eta)$ kg of gas at some temperature $T < T_1$, and therefore

$$dE_1 = d\,[c_v T G_0 (\psi - \eta)] = d\left[\frac{RT_1}{k-1} G_0 (\psi - \eta)\right]$$

In order to find the amount of energy dE_2 carried by the gases streaming through the opening during the interval dt we shall use the formula for the density of energy flux $\mathbf{j}$ through an arbitrary surface [2]

$$\mathbf{j} = \left(\frac{u^2}{2} + i\right)\rho\mathbf{u}$$

where $\mathbf{u}$ is velocity of the gas, ρ the density and i the enthalpy.

Since the absolute value of the vector $\mathbf{j}$ equals the amount of energy streaming per unit time through unit surface normal to the direction of the velocity $\mathbf{u}$, then it follows that

$$dE_2 = \left(\frac{u_k^2}{2} + \frac{k}{k-1} RT_k\right) G dt$$

where u_k and T_k are the velocity and the temperature of exit gases respectively. If the flow of gas takes place into a medium without a back-pressure (or a very small one) and the pressure within the container is sufficiently large to establish a supersonic flow during gas expansion (similarly as in the case of the deLaval nozzle), then, as is known [3], a critical condition of flow is being established in the exit opening.

Considering the flow of gas as quasi-stationary, i.e. as a continuous change in time of stationary states, we shall use formulas given by the stationary flow theory [3] in order to find the critical velocity of the flow u_k and the temperature T_k

$$u_k^2 = \frac{2k}{k-1}\,\frac{p}{\rho} = \frac{2k}{k-1} RT, \qquad T_k = \frac{2}{k+1} T$$

where $p(t)$, $\rho(t)$ and $T(t)$ are the pressure, the density and the temperature of gases in the container from which the flow takes place. Substituting the values of u_k and T_k found in this manner into the formula for dE_2 and observing that $G dt = G_0 d\eta$ we obtain

$$dE_2 = \frac{k}{k-1} RG_0 T d\eta$$

The change of kinetic energy of the piston including the adjacent layers of gas equals

$$dE_3 = d\left(\frac{Mv^2}{2}\right)$$

Here M is the induced mass of the piston and $v(t)$ the velocity of the piston.

Equating the input and output of energy

$$dE = dE_1 + dE_2 + dE_3$$

and introducing the dimensionless parameters: temperature $\tau = T/T_1$ and piston velocity $\nu = v/v_*$ where v_* is a characteristic velocity equal to

$$v_* = \sqrt{\frac{2G_0RT_1}{M(k-1)}} = c\sqrt{\frac{2}{k(k-1)}\,\frac{G_0}{M}} \tag{2.2}$$

and $c = \sqrt{kRT_1}$ is the velocity of sound in the gas during its adiabatic expansion at temperature T_1, we obtain the basic energy equation

$$d\psi = d\,[\tau(\psi - \eta)] + k\tau d\eta + d\,(\nu^2) \tag{2.3}$$

which can be reduced to the following form

$$(1 - \tau)\,d\psi = (\psi - \eta)\,d\tau + (k - 1)\,\tau d\eta + d(\nu^2) \tag{2.4}$$

3. The basic relationships. The equation of motion of the piston reads

$$M\,\frac{dv}{dt} = Sp\,(t) \tag{3.1}$$

where S is the area of the container cross section and $p(t)$ is the pressure of gases in the container.

The amount of gas introduced into the container depends on many factors. We shall consider here the simpler but at the same time very important case when the elementary amount $G_0 d\psi$ kg of gas introduced into the container during the interval dt is proportional to the elementary pressure impulse of the gas pdt, i.e. $G_0 d\psi = Bpdt$, where B is some constant. Consequently

$$G_0\,\frac{d\psi}{dt} = Bp\,(t) \tag{3.2}$$

Eliminating from equations (3.1) and (3.2) the elementary impulse pdt, and after integration, we obtain the velocity of the piston

$$v = \frac{SG_0}{MB}\,(\psi - \psi_0) \tag{3.3}$$

where ψ_0 is the fraction of gas introduced into the container at the moment when the piston begins to move. The time rate of the gas flow $G(t)$ corresponding to the critical condition of flow can be expressed on the basis of stationary flow theory [3] in the form

$$G\,(t) = D\,\frac{p\,(t)}{\sqrt{\tau\,(t)}} \tag{3.4}$$

where the constant D has the following value

$$D = \left(\frac{2}{k+1}\right)^{\frac{k+1}{2(k-1)}} S_k \sqrt{\frac{k}{RT_1}} \tag{3.5}$$

and S_k is the cross section of the exit opening.

Using the above values for $G(t)$ and the formulas (2.1) and (3.4) we find

$$\frac{d\eta}{dt} = \frac{D}{G_0}\,\frac{p\,(t)}{\sqrt{\tau\,(t)}} \tag{3.6}$$

and from the system of formulas (3.2) and (3.6) after elimination of the elementary impulse pdt we obtain

$$\frac{d\eta}{d\psi} = \frac{D}{B}\,\frac{1}{\sqrt{\tau}}, \quad \text{or} \quad \tau = \left(\frac{D}{B}\right)^2 \left(\frac{d\psi}{d\eta}\right)^2 \tag{3.7}$$

Substituting the found expression for r into equation of energy (2.4) and using the relationship (3.3), after some simplification and dividing all the members by a factor $d\psi/d\eta \neq 0$ we obtain the following nonlinear differential equation of the second order

$$(\psi - \eta)\frac{d^2\psi}{d\eta^2} + A_1\frac{d\psi}{d\eta} + A_2\left(\frac{d\psi}{d\eta}\right)^2 + A_0 + A(\psi - \psi_0) = 0 \qquad (3.8)$$

Here

$$A_0 = -\frac{1}{2}\left(\frac{B}{D}\right)^2, \qquad A_1 = \frac{k-1}{2}, \qquad A_2 = \frac{1}{2} \qquad (3.9)$$

$$A = \frac{k-1}{2k}\left(\frac{k+1}{2}\right)^{\frac{k+1}{k-1}} \frac{G_0}{M}\left(\frac{S}{S_k}\right)^2 \qquad (3.10)$$

We shall determine the initial conditions from the fact that the moment at which the gas begins to flow out of the container coincides with the beginning of the movement of the piston, and the temperature of the gas equals at this time $T = T_1$, i.e. for $\eta = 0$ we have $\psi = \psi_0$ and $r = 1$. Consequently on the basis of (3.7)

$$\left(\frac{d\psi}{d\eta}\right)_{\eta=0} = \frac{B}{D}$$

The variables ψ and η vary in limits $\psi_0 \leqslant \psi \leqslant 1$ ($\psi = 1$ corresponds to the end of gas supply to the container) and $0 \leqslant \eta \leqslant \eta_1$, (the value η_1 corresponds to the time when $\psi = 1$).

We shall consider various cases of integrating the equation (3.8).

4. The flow of gas out of the container having constant volume. One obtains this condition when the piston, which is formed by the rear wall of the container, is stationary. This of course can be attained by assuming that the mass of the piston is infinite, i.e. $G_0/M = 0$. In this case the parameter A (3.10) becomes zero and the basic equation (3.8) has the following form

$$(\psi - \eta)\frac{dx}{d\eta} + A_2x^2 + A_1x + A_0 = 0 \qquad \left(x = \frac{d\psi}{d\eta}\right) \qquad (4.1)$$

Equation (4.1) can be integrated by means of the following substitution

$$(\psi - \eta)\,\varphi(x) = c \qquad (4.2)$$

Taking derivative with respect to η we obtain

$$(x-1)\,\varphi(x) + (\psi - \eta)\frac{d\varphi}{dx}\frac{dx}{d\eta} = 0$$

Substituting now for $(\psi - \eta)dx/d\eta$ from equation (4.1) we obtain, after separation of variables, the following relation for the function $\phi(x)$:

$$\frac{d\varphi}{\varphi} = \frac{(x-1)\,dx}{A_0 + A_1x + A_2x^2} = \frac{2\,(x-1)\,dx}{(x-x_1)(x-x_2)} \qquad (4.3)$$

where the notation has been introduced:

$$x_{1,2} = -\frac{k-1}{2} \pm \sqrt{\left(\frac{k-1}{2}\right)^2 + \left(\frac{B}{D}\right)^2}$$

Integrating (4.3) and substituting the found expression for $\phi(x)$ into (4.2) we obtain

$$(\psi - \eta)(x - x_1)^{\alpha}(x - x_2)^{\beta} = c \qquad \left(\alpha = \frac{2(x_1 - 1)}{x_1 - x_2}, \ \beta = \frac{2(x_2 - 1)}{x_2 - x_1}\right) \tag{4.4}$$

Constant c is determined from the condition that at the moment the flow starts $\psi = \psi_0$, $\eta = 0$, and $x = x_0$ where

$$x_0 = \left(\frac{d\psi}{d\eta}\right)_{\eta=0} = \frac{B}{D} \tag{4.5}$$

The equation of state $pw = RT$ will be used to determine the gas pressure $p(t)$. Since $T = T_1\tau$, and the specific volume $w = W_0/G_0(\psi - \eta)$, where W_0 is the volume of the container from which the flow issues, then

$$p(t) = \frac{RT_1}{W_0} G_0 \tau (\psi - \eta)$$

Using the expression $(\psi - \eta)$ determined from (4.4) and noting that the dimensionless temperature on the basis of (3.7) and (4.5) is

$$\tau = \left(\frac{D}{B}\right)^2 \left(\frac{d\psi}{d\eta}\right)^2 = \left(\frac{x}{x_0}\right)^2 \tag{4.6}$$

we obtain the following from the state equation $pw = RT$

$$p(x) = c_1 x^2 (x - x_1)^{-\alpha} (x - x_2)^{-\beta} \left(c_1 = c\frac{RT_1}{W_0}\frac{G_0}{x_0^2}\right) \tag{4.7}$$

We shall discuss the condition of the extreme value of pressure within the container. Taking the logarithmic derivative of $p(x)$ and using the values of α, β, x_1, x_2 and x_0 we obtain

$$\frac{dp}{dx} = 2kp(x)\frac{x - k^{-1}x_0^2}{x(x - x_1)(x - x_2)}$$

from which follows that for the value of $x = k^{-1}x_0^2$ the pressure of gases in the container will attain an extreme value, since it can be shown that the second derivative d^2p/dx^2 is different from zero at $x = k^{-1}x_0^2$. The analysis of expression d^2p/dx^2 at $x = k^{-1}x_0^2$ shows that its sign depends on the sign of expression $\lambda = (k^{-1}x_0^2 - 1)$; thus: 1) for $\lambda > 0$, corresponding to the value $x_0 > \sqrt{k}$, the gas pressure is a minimum, 2) for $\lambda < 0$, corresponding to $x_0 < \sqrt{k}$, the gas pressure is a maximum. If $\lambda = 0$, which corresponds to the case $x_0 = \sqrt{k}$, the problem needs an additional investigation. In this case, as will be shown later, a stationary process of gas outflow takes place at a constant dimensionless temperature $\tau = k^{-1}$; the gas pressure in this case weill remain constant.

5. The stationary flow of gas from the container. Suppose, starting at a certain instant, the dimensionless inflow of gas $d\psi$ equals the dimensionless outflow $d\eta$ and consequently

$$x = \frac{d\psi}{d\eta} = 1$$

From relations (3.7) and (4.5)

$$\tau = \left(\frac{D}{B}\right)^2 = \frac{1}{x_0^2} \tag{5.1}$$

and as a result the expansion of gas in the container will be isothermal.

Considering now equation (3.8) and noting that $d\psi/d\eta = 1$ and $d^2\psi/d\eta^2 = 0$ we obtain

$$A_0 + A_1 + A_2 + A(\psi - \psi_0) = 0$$

It follows from the above that the constant A (3.10) should be zero and therefore

$$A_0 + A_1 + A_2 = 0 \tag{5.2}$$

The fact that parameter A (3.10) vanishes reduces to the condition $G_0/M = 0$ and this corresponds to the stationary rear wall of the container (since the mass M of the piston becomes infinite in this case). The result thus obtained indicates that the stationary process of flow here analyzed can occur only from a container having a constant volume.

Substituting into equation (5.2) the values of coefficients A_i (3.9) and using relations (5.1) we find that $r = k^{-1}$, i.e. the ratio of the gas temperature T within the container to the temperature T_1 of gas entering the container remains constant and equal to the ratio of specific heats c_p/c_v. This result is known as Langevin's theorem [4, 5]

It should be pointed out that the Langevin formula $r = k^{-1}$ may be also obtained starting from the energy equation (2.4), if one assumes that $d(\nu^2) = 0$, $dr = 0$, and $d\psi = d\eta$. This yields $(1 - kr)d\psi = 0$, from which one obtains the Langevin formula.

6. The flow of gas from a container with a moving wall. We shall consider the flow of gas out of a container with a moving wall from the instant when the supply of gas to the container is being stopped; this corresponds to the value of dimensionless variable $\psi = 1$. The corresponding energy equation is obtained if in equation (2.4) we put $\psi = 1$ and $d\psi = 0$.

This yields

$$(1 - \eta)\, d\tau + (k - 1)\, \tau\, d\eta + d(\nu^2) = 0 \tag{6.1}$$

From the system of equations (3.1) and (3.6) through elimination of the elementary impulse pdt we obtain

$$\frac{dv}{d\eta} = \frac{G_0 S}{MD} \sqrt{\tau}$$

Using the dimensionless velocity $\nu = v/v_*$, the dimensionless temperature τ can be determined as

$$\tau = \frac{1}{A}\left(\frac{d\nu}{d\eta}\right)^2 \qquad (6.2)$$

where the constant A is defined as before by expression (3.10). Substituting the expression (6.2) for τ into the energy equation (6.1) we find

$$\frac{2}{A}\frac{d\nu}{d\eta}\left\{(1-\eta)\frac{d^2\nu}{d\eta^2} + \frac{k-1}{2}\frac{d\nu}{d\eta} + A\nu\right\} = 0$$

Dividing by the factor $d\nu/d\eta \neq 0$ and introducing a new variable $y = 1 - \eta$, we obtain

$$y\frac{d^2\nu}{dy^2} - \frac{k-1}{2}\frac{d\nu}{dy} + A\nu = 0 \qquad (6.3)$$

It can be directly verified that this equation can be reduced by the change of variables

$$y = \frac{1}{4A}Z^2, \qquad \nu(y) = Z^n W(Z) \qquad (6.4)$$

where the constant $n = ½(k + 1)$, to the following Bessel equation

$$\frac{d^2 W}{dZ^2} + \frac{1}{Z}\frac{dW}{dZ} + \left(1 - \frac{n^2}{Z^2}\right)W = 0 \qquad (6.5)$$

It is known [6] that if n is not an integer, which happens to be the case in our problem, then the general integral of equation (6.5) can be expressed by means of Bessel functions of the first kind $J_n(Z)$ and $J_{-n}(Z)$:

$$W(Z) = C_1 J_n(Z) + C_2 J_{-n}(Z)$$

where C_1 and C_2 are constants determined from initial conditions.

We shall switch now from the variables Z and W to the variables y and ν for which purpose we shall use formulae (6.4). Incorporating constant multipliers into the constants C_1 and C_2 we obtain the following expression for the dimensionless velocity ν of the piston:

$$\nu(y) = y^{1/2 n}\{C_1 J_n(2\sqrt{Ay}) + C_2 J_{-n}(2\sqrt{Ay})\} \qquad (6.6)$$

Using formula (6.2) we can determine the dimensionless temperature τ:

$$\sqrt{\tau} = \frac{1}{\sqrt{A}}\frac{d\nu}{d\eta} = -\frac{1}{\sqrt{A}}\frac{d\nu}{dy}$$

If now by means of (6.6) one determines the derivative $d\nu/dy$ and then uses the well known relationships [6]

$$2J_n'(Z) = J_{n-1}(Z) - J_{n+1}(Z), \qquad \frac{2n}{Z}J_n(Z) = J_{n-1}(Z) + J_{n+1}(Z)$$

one obtains after some simplifications the following expression for τ:

$$\sqrt{\tau} = y^{\frac{n-1}{2}}\{-C_1 J_{n-1}(2\sqrt{Ay}) + C_2 J_{-n+1}(2\sqrt{Ay})\} \qquad (6.7)$$

Equations (6.6) and (6.7) fully solve the problem. In order to determine the constants C_1 and C_2 one has to make use of initial conditions: when $\eta = \eta_1$, then $\tau = \tau_1$ and $\nu = \nu_1$.

The formulae obtained above can be transformed if expansions of Bessel functions are used:

$$J_n(Z) = \sum_{m=0}^{\infty} \frac{(-1)^m (1/2\, Z)^{n+2m}}{m!\,\Gamma(-n+m+1)}$$

Substituting this expansion into (6.6) and (6.7) and observing that $Z = 2\sqrt{Ay}$, after some simplification we obtain:

$$\nu(y) = C_1{}^* \Phi_1(A, n, y) + C_2{}^* \Phi_2(A, n, y) \tag{6.8}$$

$$\sqrt{\tau} = -\frac{1}{\sqrt{A}} \{C_1{}^* R_1(A, n, y) + C_2{}^* R_2(A, n, y)\} \tag{6.9}$$

where

$$C_1{}^* = C_1 \frac{A^{1/2\, n}}{\Gamma(n+1)}, \qquad C_2{}^* = C_2 \frac{A^{-1/2\, n}}{\Gamma(-n+1)}$$

and the functions $\Phi_1(A,n,y)$, $\Phi_2(A,n,y)$, $R_1(a,n,y)$ and $R_2(A,n,y)$ have the following expansions

$$\Phi_1(A, n, y) = y^n \Big(1 + \sum_{m=1}^{\infty} A_{m,-n} y^m\Big), \qquad \Phi_2(A, n, y) = 1 + \sum_{m=1}^{\infty} A_{m,n} y^m$$

$$R_1(A, n, y) = y^{n-1} \Big\{n + \sum_{m=1}^{\infty} (n+m)\, A_{m,-n} y^m\Big\}, \qquad R_2(A, n, y) = \sum_{m=1}^{\infty} m A_{m,n} y^{m-1}$$

The coefficients $A_{m,n}$ and $A_{m,-n}$ entering into these expansions are determined by means of the following recurrence formulas

$$A_{m+1,n} = \frac{-A}{(m+1)(m-n+1)} A_{m,n}, \qquad A_{1n} = \frac{A}{n-1}$$

Coefficients $A_{m,-n}$ are obtained from $A_{m,n}$ by means of changing n into $-n$.

The gas pressure $p(t)$ and also the piston trajectory can be found by assuming an adiabatic expansion of gas and by using the Poisson adiabatic equation $pw^k = p_1 w_1{}^k$ where p_1 and w_1 are respectively the pressure and specific volume of gases within the container at the moment when the process begins.

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Translated by B.Z.